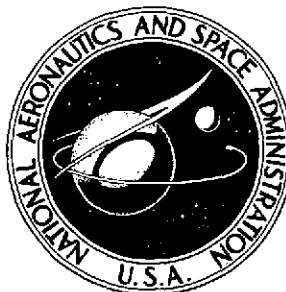


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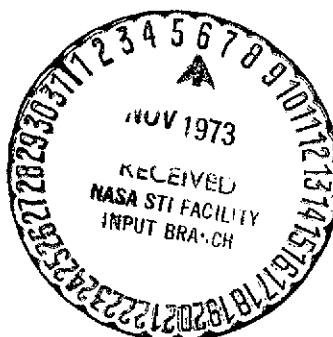
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MODELING OF TURBULENT TRANSPORT IN THE SURFACE LAYER

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SUMMARY

The turbulence equations as written by Donaldson using the method of invariant modeling have been applied to the following limiting cases of the surface or constant flux layer of the planetary boundary layer:

- (1) Neutrally stable
- (2) Stable (above influence of surface roughness)
- (3) Nearly neutrally stable
- (4) Very unstable (free convection)

For the neutrally stable case, the equations are shown to admit as a solution the familiar logarithmic profile. By use of this result, boundary conditions suitable for the surface layer are defined and are simple to apply to rough surfaces.

Expressions for the macroscale length Λ are given for each case. The parameters b , relating the microscale to the macroscale, and c , the ratio of macroscale length to height, are computed to fit atmospheric data. Owing to the structural difference between mechanically and thermally produced turbulence, different values are found for b and c for the neutrally stable and unstable cases. The b -value for the stable case agrees very closely with that for the neutrally stable case.

INTRODUCTION

The most studied case of atmospheric turbulence is that within the surface layer, which has a thickness on the order of 20 to 200 meters (ref. 1, p. 100). Many data have been taken by use of instrumented masts, and continue to be taken as more sophisticated instrumentation becomes available. These data have spurred the development of a body of theory, which may be used for analysis of data or for prediction of turbulent transport for given meteorological conditions. The basis of much of this theory is the Monin-Obukhov similarity theory (ref. 2). By this theory a characteristic velocity, length, and temperature can be defined such that all quantities when nondimensionalized by using these three parameters are reduced to a set of universal curves, which have been empirically defined (refs. 3 and 4).

The surface layer is dependent upon the roughness of the surface beneath it, which, for example, may be grass, crops, trees, or buildings. This effect is accounted for by use of the roughness length, which is found empirically and is tabulated for various surfaces (refs. 5, 6, and 7).

One approach to the computation of atmospheric turbulence is that of Donaldson and others (refs. 8 and 9), who modeled second-order correlations of the Navier-Stokes equations. Results are presented for application to the surface layer in reference 10. In reference 10, the boundary conditions suitable for a flat plate were used; that is, velocity and turbulence vanished at the surface. Also, the mean velocity and temperature profiles were specified (by using data from the work described in ref. 4), and the profiles of turbulent quantities were computed. Donaldson's equations require the specification of a macroscale mixing length. This length is typically taken to be proportional to the distance from the surface out to some point, which must be selected, beyond which the macroscale length is assumed constant.

This paper presents a study of the application of Donaldson's equations to the surface layer in light of existing surface-layer theory. This approach gives a guide to the application of Donaldson's equations to atmospheric problems and conversely provides a possibility of generalizing surface-layer theory to a broader class of problems. The approach used in references 8 to 10 is to develop a single comprehensive theory of turbulence with a single set of constants, which will be applicable to all cases, from turbulent flow over a flat plate in a wind tunnel to atmospheric turbulence. The philosophy of the present paper is that atmospheric turbulence is sufficiently important to warrant computation of a set of constants specifically for this case. In this paper, Donaldson's equations are simplified for the large Reynolds numbers characteristic of atmospheric turbulence. Also, the turbulent Prandtl number is introduced into the model. The boundary conditions suitable for a rough surface (e.g., grass, forest, or city) are considered, and heuristic arguments are given for a set, which is selected. The equations are then solved for a neutrally stable atmosphere (no vertical heat flux), thereby resulting in the familiar logarithmic wind profile. This serves three purposes: The suitability of the governing equations is verified; the suitability of the boundary conditions is verified; and two constants in the method are evaluated for atmospheric application. Next the case of stable stratification above the influence of the surface roughness is treated. From this study comes a mixing length appropriate to the stable layer. This length can be used to determine the point above which the mixing length becomes constant. Finally, the free-convection or unstable case, in which turbulence is thermally produced, is considered.

SYMBOLS

a	constant relating macroscale of turbulence to microscale (see eq. (10))
A_n	set of coefficients in series solution of temperature equation (see eq. (62a))
b	constant relating macroscale of turbulence to microscale (see eq. (10))
B	constant in S_{33} -equation for free convection (see eq. (94))
B_n	set of coefficients in series solution of temperature variance equation (see eq. (62b))
c	constant relating macroscale length Λ to height (see eq. (39))
C_n	set of coefficients in series solution of horizontal heat-flux density equation (see eq. (62c))
F_i	set of coefficients in free-convection solution (see eqs. (77) to (86))
g	gravitational acceleration, m/sec^2
H	constant for free-convection case defined by equation (100)
k	von Kármán constant
K	twice total turbulent kinetic energy per mass, $S_{11} + S_{22} + S_{33}$, m^2/sec^2
L	Obukhov length (see eq. (23)), m
L_2	length appearing in analysis of nearly neutral atmosphere (see eq. (63)), m
n	index in series solutions
Pr	turbulence Prandtl number
$q_i = \overline{T'(v_i - \bar{v}_i)}$	(proportional to heat flux in i -direction), $m-K/sec$
Q	vertical heat flux (positive upward), $kg-K/sec-m^2$
r	temperature variance, $\overline{T'^2}$, K^2

R_Λ	turbulence Reynolds number (see eq. (11))
s	negative of turbulent shear stress, $\text{kg/sec}^2\text{-m}$
S_{ij}	turbulent stress tensor, $\overline{(v_i - \bar{v}_i)(v_j - \bar{v}_j)}$, m^2/sec^2
t	time, sec
T	mean temperature, K
T'	temperature fluctuation from mean, K
T_o	mean temperature at z_o
T_*	scaling temperature (see eq. (48)), K
u	mean velocity in direction of mean flow, m/sec
u_*	friction velocity, m/sec
v_i	velocity component in i-direction, m/sec
w	vertical component of velocity, m/sec
x	coordinate in direction of mean flow, m
z	coordinate in vertical direction, m
z_o	roughness height, m
$a = (q_3/S_{13})\left(\frac{du}{dz}/\frac{dT}{dz}\right)$	
β	constant in solution form for free convection (see eqs. (72))
$\delta = 1 + 2b - \frac{2}{3} c^2$	
Λ	macroscale length, m

λ	microscale length, m
μ	molecular viscosity, kg/m-sec
μ_2	second viscosity coefficient, kg/m-sec
ν	exponent in equation (72)
ρ	density, kg/m ³
σ_w, σ_T	variance of w and T, respectively
ϕ_h	nondimensional temperature gradient (see eq. (58))
ϕ_m	nondimensional velocity gradient (see eq. (57))

The bar (—) over a symbol indicates time averaged value.

GOVERNING EQUATIONS

A set of equations for turbulent flow applicable to atmospheric motions is given in reference 8, along with a suggested modeling of such correlation terms as is required to close the set of equations. In reference 9 these equations with this modeling are applied to the case of steady-state parallel turbulent shearing flow in the atmosphere over a large uniform plane. For this case, the flow is a function only of the vertical coordinate z and the continuity equation is trivially satisfied. With no body force or radiative heat source, the remaining equations reduce to

Momentum:

$$\mu \frac{d^2 \bar{u}}{dz^2} - \frac{d}{dz} (\rho S_{13}) = 0 \quad (1)$$

Energy:

$$\mu \frac{d^2 \bar{T}}{dz^2} - \frac{d}{dz} (\rho q_3) = 0 \quad (2)$$

Longitudinal turbulent energy:

$$\begin{aligned}
 & -2\rho S_{13} \frac{du}{dz} + \frac{d}{dz} \left(\rho \sqrt{K} \Lambda \frac{dS_{11}}{dz} \right) \\
 & - \rho \frac{\sqrt{K}}{\Lambda} \left(S_{11} - \frac{1}{3} K \right) + \mu \frac{d^2 S_{11}}{dz^2} - \frac{2\mu}{\lambda^2} S_{11} = 0
 \end{aligned} \tag{3}$$

Lateral turbulent energy:

$$\frac{d}{dz} \left(\rho \sqrt{K} \Lambda \frac{dS_{22}}{dz} \right) - \rho \frac{\sqrt{K}}{\Lambda} \left(S_{22} - \frac{1}{3} K \right) + \mu \frac{d^2 S_{22}}{dz^2} - \frac{2\mu}{\lambda^2} S_{22} = 0 \tag{4}$$

Vertical turbulent energy:

$$\begin{aligned}
 & 5 \frac{d}{dz} \left(\rho \sqrt{K} \Lambda \frac{dS_{33}}{dz} \right) + 2 \frac{d\rho}{dz} \left(\Lambda \sqrt{K} \frac{dS_{33}}{dz} \right) - \rho \frac{\sqrt{K}}{\Lambda} \left(S_{33} - \frac{1}{3} K \right) \\
 & + \mu \frac{d^2 S_{33}}{dz^2} - \frac{2\mu}{\lambda^2} S_{33} + \frac{2g}{T} \rho q_3 - 2(\mu + \mu_2) S_{33} \left[\frac{d}{dz} \left(\frac{1}{\rho} \frac{d\rho}{dz} \right) \right. \\
 & \left. - \frac{1}{\rho^2} \left(\frac{d\rho}{dz} \right)^2 \right] = 0
 \end{aligned} \tag{5}$$

Shear stress:

$$\begin{aligned}
 & -\rho S_{33} \frac{du}{dz} + 3 \frac{d}{dz} \left(\rho \sqrt{K} \Lambda \frac{dS_{13}}{dz} \right) - \rho \frac{\sqrt{K}}{\Lambda} S_{13} + \Lambda \sqrt{K} \frac{dS_{13}}{dz} \frac{d\rho}{dz} \\
 & + \mu \frac{d^2 S_{13}}{dz^2} - \frac{2\mu}{\lambda^2} S_{13} - (\mu + \mu_2) S_{13} \left[\frac{d}{dz} \left(\frac{1}{\rho} \frac{d\rho}{dz} \right) - \frac{1}{\rho^2} \left(\frac{d\rho}{dz} \right)^2 \right] + \frac{\rho g}{T} q_1 = 0
 \end{aligned} \tag{6}$$

Longitudinal heat flux:

$$-\rho S_{13} \frac{dT}{dz} - \rho q_3 \frac{du}{dz} + \frac{d}{dz} \left(\frac{\rho \sqrt{K} \Lambda}{Pr} \frac{dq_1}{dz} \right) - \rho \frac{Pr \sqrt{K}}{\Lambda} q_1 + \mu \frac{d^2 q_1}{dz^2} - \frac{2\mu}{\lambda^2} q_1 = 0 \quad (7)$$

Vertical heat flux:

$$\begin{aligned} & -\rho S_{33} \frac{dT}{dz} + 3 \frac{d}{dz} \left(\frac{\rho \sqrt{K} \Lambda}{Pr} \frac{dq_3}{dz} \right) - \rho \frac{Pr \sqrt{K}}{\Lambda} q_3 + \mu \frac{d^2 q_3}{dz^2} - \frac{2\mu}{\lambda^2} q_3 + \frac{\rho g}{T} r \\ & - (\mu + \mu_2) q_3 \left[\frac{d}{dz} \left(\frac{1}{\rho} \frac{d\rho}{dz} \right) - \frac{1}{\rho^2} \left(\frac{d\rho}{dz} \right)^2 \right] = 0 \end{aligned} \quad (8)$$

Temperature variance:

$$-2\rho q_3 \frac{dT}{dz} + \frac{d}{dz} \left(\rho \frac{\sqrt{K} \Lambda}{Pr} \frac{dr}{dz} \right) + \mu \frac{d^2 r}{dz^2} - \frac{2\mu}{\lambda^2} r = 0 \quad (9)$$

In these equations appear two lengths, Λ and λ , which are the macroscale and microscale, respectively, of the problem. For turbulent transfer of heat terms in the present paper, the mixing length is divided by the turbulent Prandtl number Pr . The introduction of this modification to the original model of references 8 to 10 is in analogy with laminar transport equations. In the model of references 8 to 10, it is assumed that

$$\lambda = \frac{\Lambda}{\sqrt{a + bR_\Lambda}} \quad (10)$$

where

$$R_\Lambda = \frac{\rho \sqrt{K} \Lambda}{\mu} \quad (11)$$

Equation (10) was assumed in references 8 to 10 because it behaves for large and small R_Λ as desired and permits self-similar solutions of the decay of a free jet. Also, $a = 2.5$ and $b = 0.125$ were selected in references 8 to 10, based on extensive wind-tunnel data. In the present paper, the value of b applicable to atmospheric problems is considered. For atmospheric processes, turbulence Reynolds number $bR_\Lambda \gg a$, and R_Λ is very high so that

$$\frac{2\mu}{\lambda^2} \approx 2b \frac{\rho \sqrt{K}}{\Lambda} \quad (12)$$

Where Λ appears as a mixing length for temperature or a correlation involving temperature, it is divided by Pr . Equation (12) is substituted into equations (3) to (9), and the remaining terms containing μ , but not containing λ^2 , are neglected as these terms are for molecular transport, which is much smaller than turbulent transport in the atmosphere. The momentum and energy equations reduce to

$$\frac{d}{dz} (\rho S_{13}) = 0$$

$$\frac{d}{dz} (\rho q_3) = 0$$

and are integrated to yield

$$\rho S_{13} = \text{Constant} = -s = -\rho u_*^2 \quad (13)$$

$$\rho q_3 = \text{Constant} = Q \quad (14)$$

where u_* is the friction velocity. These are the familiar results that shear stress and heat flux are constant through the surface layer. Gradients of density are neglected. Equations (3) to (9) now reduce to

Temperature variance:

$$\frac{d}{dz} \left(\rho \frac{\sqrt{K}\Lambda}{\text{Pr}} \frac{dr}{dz} \right) - 2b \frac{\text{Pr} \rho \sqrt{K}}{\Lambda} r = 2Q \frac{dT}{dz} \quad (15)$$

Heat flux:

$$x: \quad \frac{d}{dz} \left(\frac{\rho \sqrt{K}\Lambda}{\text{Pr}} \frac{dq_1}{dz} \right) - (1 + 2b) \frac{\text{Pr} \rho \sqrt{K}}{\Lambda} q_1 = -s \frac{dT}{dz} + Q \frac{du}{dz} \quad (16)$$

$$z: \quad -\frac{\text{Pr} \sqrt{K}}{\Lambda} (1 + 2b) Q = \rho S_{33} \frac{dT}{dz} - \frac{\rho g}{T} r \quad (17)$$

Covariance tensor:

$$S_{11}: \quad \frac{d}{dz} \left(\rho \sqrt{K}\Lambda \frac{dS_{11}}{dz} \right) - 2b \frac{\rho \sqrt{K}}{\Lambda} S_{11} = \frac{\rho \sqrt{K}}{\Lambda} \left(S_{11} - \frac{1}{3} K \right) - 2s \frac{du}{dz} \quad (18)$$

$$S_{22}: \quad \frac{d}{dz} \left(\rho \sqrt{K} \Lambda \frac{dS_{22}}{dz} \right) - 2b \frac{\rho \sqrt{K}}{\Lambda} S_{22} = \frac{\rho \sqrt{K}}{\Lambda} \left(S_{22} - \frac{1}{3} K \right) \quad (19)$$

$$S_{33}: \quad 5 \frac{d}{dz} \left(\rho \sqrt{K} \Lambda \frac{dS_{33}}{dz} \right) - 2b \frac{\rho \sqrt{K}}{\Lambda} S_{33} = \frac{\rho \sqrt{K}}{\Lambda} \left(S_{33} - \frac{1}{3} K \right) - \frac{2gQ}{T} \quad (20)$$

$$S_{13}: \quad \rho S_{33} \frac{du}{dz} - (1 + 2b) \frac{\sqrt{K}}{\Lambda} s - \frac{\rho g}{T} q_1 = 0 \quad (21)$$

BOUNDARY CONDITIONS

In classical aerodynamics, flow over a smooth surface is considered for which the boundary conditions are that all components of velocity vanish. This applies not only to the mean velocity, but as well to the turbulent part. The vanishing of the turbulence at the surface requires that momentum transport be carried by a laminar sublayer. For the case of the atmospheric surface layer, the surface may be smooth, as a snow-covered plain, but is typically rough, as a forest, a city, or crops in a field. Although each leaf in the forest may have a laminar sublayer, there is a lack of enthusiasm for considering this level of detail in specifying the geometry. Also, much of the momentum transfer at the rough surface is caused by the drag of the wind on trees or houses.

For a neutrally stable surface layer, that is when $Q = 0$, many studies have shown that the surface-layer wind profile above the roughness elements may be described by a logarithmic function

$$u = \frac{u_*}{k} \ln \left(\frac{z}{z_0} \right) \quad (22)$$

where z_0 is the roughness height characterizing the surface (refs. 1, 5, and 6). This relation is written such that $u = 0$ at $z = z_0$. The roughness height z_0 is tabulated for different types of surfaces on page 233 of reference 5 and page 150 of reference 6.

The boundary conditions for the turbulence terms S_{11} , S_{22} , S_{33} , and r are arrived at by use of the Monin-Obukhov similarity theory together with data. In references 1 and 3, it is pointed out that according to Monin-Obukhov theory, S_{11}/S_{13} , S_{22}/S_{13} , S_{33}/S_{13} , and r/T_*^2 are functions only of z/L , where L is the Obukhov length given by

$$L = - \frac{u_*^3 \rho T}{kgQ} \quad (23)$$

and T_* is the scaling temperature given by

$$T_* = - \frac{Q}{ku_* \rho}$$

Because u_* and Q are constant through the surface layer, L and T_* are each constant. These functions are defined from data in chapter 4 of reference 1 and in reference 4, and it is pointed out that S_{11} , S_{22} , and S_{33} vary very slowly with altitude and are moreover constant for the neutrally stable surface layer. On this basis, the boundary conditions, which are proposed in this paper for these quantities at the rough surface, are

$$\frac{dS_{11}}{dz} = \frac{dS_{22}}{dz} = \frac{dS_{33}}{dz} = 0 \quad (z = z_0) \quad (24)$$

These conditions state that near a rough surface turbulence is generated at the same rate as it is dissipated. Use of these conditions assures that the surface layer will approach neutrally stable layer behavior.

From the discussion of temperature variance in chapter 4 of reference 1, \sqrt{r}/T_* is constant with altitude for an unstable surface layer for small z/L and for a nearly neutral surface layer. Also, for small z/L in a stable surface layer, $d(\sqrt{r}/T_*)/dz = 0$. Thus the boundary condition to be used herein is

$$\frac{dr}{dz} = 0 \quad (z = z_0) \quad (25)$$

Equations (15), (16), (18), (19), and (20) are second order and as such require two boundary conditions each. The form of these equations, taken one at a time, is such that there is a homogeneous solution which vanishes exponentially with altitude and one which grows exponentially with altitude. For these equations, the exponentially increasing term is eliminated. A situation where turbulence is generated above the surface layer can be conceived. This turbulence would be damped exponentially as it diffused downward. By using boundary conditions, which get rid of the exponentially increasing part of the solution, this situation is omitted.

One more boundary condition needed for the longitudinal heat-flux equation is that the q_1 -profile be well behaved at the rough surface.

Note that, had viscous terms been retained, the surface would not be a singular point of the differential equations governing q_1 . There would then be no occasion to require that

the solution be well behaved, but there would be the laminar boundary condition that $q_1 = 0$ at $z = z_0$.

NEUTRALLY STABLE ATMOSPHERE

The neutrally stable atmosphere is now considered. The purpose of this is not to present another derivation of the familiar logarithm profile but to verify the application of the equations of reference 8 and the use of the boundary conditions proposed here for analysis of the surface layer. Also, a reevaluation of some arbitrary constants in the model is desired for atmospheric use.

The neutrally stable atmosphere is defined by the vertical heat flux $Q = 0$. A solution to equations (15) to (17) is that r , q , and dT/dz vanish throughout the surface layer. Equations (18) to (21) are solved by assuming that throughout the surface layer

$$\frac{dS_{11}}{dz} = \frac{dS_{22}}{dz} = \frac{dS_{33}}{dz} = 0 \quad (26)$$

so that these equations reduce to

$$-2b\rho\sqrt{K} S_{11} = \rho\sqrt{K} \left(S_{11} - \frac{1}{3} K \right) - 2s\Lambda \frac{du}{dz} \quad (27)$$

$$-2b\rho\sqrt{K} S_{22} = \rho\sqrt{K} \left(S_{22} - \frac{1}{3} K \right) \quad (28)$$

$$-2b\rho\sqrt{K} S_{33} = \rho\sqrt{K} \left(S_{33} - \frac{1}{3} K \right) \quad (29)$$

Summing equations (27) to (29) and substituting $K = S_{11} + S_{22} + S_{33}$ yields

$$b\rho K^{3/2} = s\Lambda \frac{du}{dz} \quad (30)$$

Equations (28) and (29) give

$$S_{22} = S_{33} = \frac{K}{3(1 + 2b)} \quad (31)$$

From equations (27) and (30)

$$S_{11} = \frac{1 + 6b}{3(1 + 2b)} K \quad (32)$$

Equations (30), (31), and (21) are combined to produce the result

$$K = \sqrt{\frac{3}{b}} (1 + 2b) \frac{s}{\rho} \quad (33)$$

The friction velocity u_* for turbulent flow is defined by

$$\rho u_*^2 = s \quad (34)$$

Equations (31), (32), and (33) thus yield

$$\frac{S_{22}}{u_*^2} = \frac{S_{33}}{u_*^2} = \frac{1}{\sqrt{3b}} \quad (35)$$

$$\frac{S_{11}}{u_*^2} = \frac{1 + 6b}{\sqrt{3b}} \quad (36)$$

$$\frac{K}{u_*^2} = \frac{3(1 + 2b)}{\sqrt{3b}} \quad (37)$$

Equation (30) becomes

$$\Lambda \frac{du}{dz} = 3^{3/4} b^{1/4} (1 + 2b)^{3/2} u_* \quad (38)$$

At this point, the mixing length Λ must be specified. In reference 8 and elsewhere, the form assumed for Λ is

$$\Lambda = cz \quad (39)$$

Thus the solution to equation (38) is.

$$u = \left[\frac{3^{3/4} b^{1/4} (1 + 2b)^{3/2}}{c} \right] u_* \ln \frac{z}{z_0}$$

Note that the roughness height is thus incorporated into the solution, as typically is done in surface-layer theory. Comparison of this solution with equation (22) shows that

$$\frac{3^{3/4} b^{1/4} (1 + 2b)^{3/2}}{c} = \frac{1}{k} \quad (40)$$

The results obtained are consistent with the initial assumption of equation (26) and satisfy the governing equations and boundary conditions; hence, the solution is self consistent.

The theory is now compared with empirical results. The figures in reference 3 show that for a neutral atmosphere

$$\frac{\sqrt{S_{11}}}{u_*} = 2.6$$

$$\frac{\sqrt{S_{22}}}{u_*} = 2$$

$$\frac{\sqrt{S_{33}}}{u_*} = 1.3$$

From these values

$$\frac{K}{u_*^2} = 12.45$$

Equation (37) may now be solved for b , thus giving $b = 0.021$. Equations (35) and (36) next give

$$\frac{\sqrt{S_{11}}}{u_*} = 2.12$$

$$\frac{\sqrt{S_{22}}}{u_*} = 2$$

$$\frac{\sqrt{S_{33}}}{u_*} = 2$$

By assuming von Kármán's constant $k = 0.35$ for atmospheric application (ref. 4), equation (4) gives

$$c = 0.323$$

As applied, the present model is seen to predict a longitudinal turbulent energy which is lower and a vertical turbulent energy which is higher than data. Fortuitously, the lateral turbulent energy is in agreement. The arbitrariness of the results is apparent; equation (35) or (36) could have been used to evaluate b , thereby changing the numerical results. Equation (37) is solved for b because the total-turbulent-energy term K is felt to be more representative than any one of the three parts S_{11} , S_{22} , or S_{33} . Also, K is the only one of these quantities to appear in each equation.

Thus far, it has been shown that the model developed by Donaldson, Sullivan, and Rosenbaum where used with the boundary conditions previously discussed does describe the neutrally stable boundary layer, if the partitioning of the turbulent energy among the longitudinal and vertical components is of no concern. This model uses a single microscale length λ to describe the dissipation of turbulent energy. In reference 11, Donaldson begins with three microscale lengths to describe the dissipation. In reference 8, the microscale length relates one second-order tensor to another. The simplest choice is to let the microscale length be a scalar, which permits only a single length. In order to incorporate more than a single length in the modeling of the dissipation term of the velocity correlation, the microscale lengths must be modeled as components of a fourth-order tensor. To do so is lengthy and is not done here.

The values derived herein for b and c differ considerably from the values $b = 0.125$ and $c = 1.58$ found in reference 8. The reason for this disagreement is that the present paper uses atmospheric data to evaluate these numbers and reference 8 uses flat-plate data from reference 12.

STABLE ATMOSPHERE

A stably stratified surface layer, that is, one in which the temperature increases with height, is described quite well by a log-linear profile. Near the ground, the logarithmic term dominates. Well above the ground the velocity and temperature gradients are nearly constant, as are the turbulence and mixing length Λ . For this condition, the governing equations (15) to (21) simplify considerably, thereby reducing to

Temperature variance:

$$-2b \frac{\text{Pr} \rho \sqrt{K}}{\Lambda} r = 2Q \frac{dT}{dz} \quad (41)$$

Heat flux:

$$x: \quad -(1 + 2b) \frac{\text{Pr} \rho \sqrt{K}}{\Lambda} q_1 = -s \frac{dT}{dz} + Q \frac{du}{dz} \quad (42)$$

$$z: \quad -(1 + 2b) Q \text{Pr} \frac{\sqrt{K}}{\Lambda} = \rho S_{33} \frac{dT}{dz} - \frac{\rho g}{T} r \quad (43)$$

Covariances:

$$S_{11}: \quad -2b \frac{\rho \sqrt{K}}{\Lambda} S_{11} = \frac{\rho \sqrt{K}}{\Lambda} \left(S_{11} - \frac{1}{3} K \right) - 2s \frac{du}{dz} \quad (44)$$

$$S_{22}: \quad -2b \frac{\rho \sqrt{K}}{\Lambda} S_{22} = \frac{\rho \sqrt{K}}{\Lambda} \left(S_{22} - \frac{1}{3} K \right) \quad (45)$$

$$S_{33}: \quad -2b \frac{\rho \sqrt{K}}{\Lambda} S_{33} = \frac{\rho \sqrt{K}}{\Lambda} \left(S_{33} - \frac{1}{3} K \right) - \frac{2gQ}{T} \quad (46)$$

$$S_{13}: \quad \rho S_{33} \frac{du}{dz} - (1 + 2b) \frac{\sqrt{K}}{\Lambda} s - \frac{\rho g}{T} q_1 = 0 \quad (47)$$

Equation (44) shows that S_{11} is driven by the wind shear du/dz , and equation (46) shows that S_{33} is reduced by the downward heat flux. In this section, quantities are nondimensionalized by use of Obukhov length, scaling temperature, and friction velocity. Nondimensional quantities are denoted by a caret (^). Thus

$$\left. \begin{aligned}
 \frac{d\hat{u}}{d\hat{z}} &= \frac{L}{u_*} \frac{du}{dz} \\
 \hat{r} &= \frac{r}{T_*^2} \\
 \hat{S}_{ij} &= \frac{S_{ij}}{u_*^2} \\
 \frac{d\hat{T}}{d\hat{z}} &= \frac{L}{T_*} \frac{dT}{dz} \\
 \hat{q}_1 &= \frac{q_1}{u_* T_*}
 \end{aligned} \right\} \quad (48)$$

According to similarity theory, the nondimensionalized quantities may be expressed as functions of nondimensional height $\hat{z} = z/L$.

Equations (41) to (47) are nondimensionalized in this manner and become

Temperature variance:

$$\hat{r} = k \frac{d\hat{T}}{d\hat{z}} \frac{\hat{\Lambda}}{b \text{ Pr } \sqrt{\hat{K}}} \quad (49)$$

Heat flux:

$$x: \quad (1 + 2b) \frac{\sqrt{\hat{K}}}{\hat{\Lambda}} \hat{q}_1 \text{ Pr} = \frac{d\hat{T}}{d\hat{z}} + k \frac{d\hat{u}}{d\hat{z}} \quad (50)$$

$$z: \quad (1 + 2b) k \frac{\sqrt{\hat{K}}}{\hat{\Lambda}} \text{ Pr} = \hat{S}_{33} \frac{d\hat{T}}{d\hat{z}} - \frac{\hat{r}}{k^2} \quad (51)$$

Covariances:

$$S_{11}: \quad \hat{S}_{11} = \frac{\hat{K}}{3(1 + 2b)} + \frac{2 \hat{\Lambda}}{\sqrt{\hat{K}} (1 + 2b)} \frac{d\hat{u}}{d\hat{z}} \quad (52)$$

$$S_{22}: \quad \hat{S}_{22} = \frac{\hat{K}}{3(1 + 2b)} \quad (53)$$

$$S_{33}: \quad \hat{S}_{33} = \frac{\hat{K}}{3(1+2b)} - \frac{2\hat{\Lambda}}{(1+2b)k\sqrt{\hat{K}}} \quad (54)$$

$$S_{13}: \quad \hat{S}_{33} \frac{d\hat{u}}{d\hat{z}} - (1+2b) \frac{\sqrt{\hat{K}}}{\hat{\Lambda}} - \frac{\hat{q}_1}{k^2} = 0 \quad (55)$$

Turbulent kinetic energy:

$$b \frac{\hat{K}^{3/2}}{\hat{\Lambda}} = \frac{d\hat{u}}{d\hat{z}} - \frac{1}{k} \quad (56)$$

Equation (56) is obtained by summing equations (52), (53), and (54).

Here are eight equations in 11 unknowns: \hat{K} , \hat{S}_{11} , \hat{S}_{22} , \hat{S}_{33} , $d\hat{u}/d\hat{z}$, $d\hat{T}/d\hat{z}$, \hat{q}_1 , $\hat{\Lambda}$, \hat{r} , Pr , and b . Three additional requirements may be imposed on the problem. Two requirements are that $d\hat{u}/d\hat{z}$ and $d\hat{T}/d\hat{z}$ match empirical values. A third requirement is that \hat{K} match the empirical value.

As before, K is derived from the figures in reference 3; thus

$$\frac{\sqrt{S_{11}}}{u_*} = 2.3$$

$$\frac{\sqrt{S_{22}}}{u_*} = 2$$

$$\frac{\sqrt{S_{33}}}{u_*} = 1.3$$

or

$$\hat{S}_{11} = 5.29$$

$$\hat{S}_{22} = 4$$

$$\hat{S}_{33} = 1.69$$

and

$$\hat{K} = 10.98$$

Experimental studies (ref. 4) have shown that nondimensional wind shear

$$\phi_m = k\hat{z} \frac{d\hat{u}}{d\hat{z}}$$

and nondimensional temperature gradient $\phi_h = \hat{z} \frac{d\hat{T}}{d\hat{z}}$ are related to \hat{z} by

$$\phi_m = 1 + 4.7\hat{z} \quad (57)$$

$$\phi_h = 0.74 + 4.7\hat{z} \quad (58)$$

For $\hat{z} \gg 1$, that is, above the influence of ground roughness, these equations result in empirical values of

$$k \frac{d\hat{u}}{d\hat{z}} = 4.7 \quad (59)$$

$$\frac{d\hat{T}}{d\hat{z}} = 4.7 \quad (60)$$

For $\hat{K} = 10.98$ and $k = 0.35$, equations (59) and (56) give

$$\frac{b}{\hat{\Lambda}} = 0.29$$

thus equation (49) gives

$$\text{Pr } \hat{r} = 1.71$$

From figure 4.23 of reference 1, $\hat{r} \approx 0.09$ for the stable layer, so that for $\text{Pr} \approx O(1)$, the present equations result in a value of \hat{r} , which is high by a factor of 20. The solution to the set of equations (50) to (55) using these results is found to consist of complex numbers. Equation (49) was replaced by $\hat{r} = 0.09$, and equations (50) to (55) were solved, thereby giving

$$\hat{S}_{11} = 4.12$$

$$\hat{S}_{22} = 3.50$$

$$\hat{S}_{33} = 3.37$$

$$b = 0.0232$$

$$\hat{\Lambda} = 0.0798$$

$$Pr = 0.99$$

$$q_1 = 0.2185$$

The value of b thus computed for the stable case is quite close to the value of b computed for the neutrally stable case. This agreement increases confidence in the method. The comparison of the computed values of \hat{S}_{11} , \hat{S}_{22} , and \hat{S}_{33} with the measured values shows the agreement to be fair for \hat{S}_{11} and \hat{S}_{22} , but rather poor for \hat{S}_{33} . Because part of the \hat{S}_{22} measured is due to small-scale horizontal motions as well as mechanical turbulence and heat convection (ref. 3), the computed \hat{S}_{22} should be lower than the measured value, as it is.

Although the differences between the b -values computed here for the neutrally stable and the stable atmospheres are minute, a single b -value is desirable. This value was arrived at by slightly reposing the problem as a least squares fit. For a given b , the K , S_{11} , S_{22} , and S_{33} were computed for the neutral and for the stable atmosphere, and the sum of the squares of each of these quantities from its measured value was computed. The b for which the sum of the squares was minimum was $b = 0.0222$. This value is considered as the best fit for the two cases jointly. The corresponding mixing length in the stable case is $\hat{\Lambda} = 0.0785$.

In reference 4, $a = \frac{q_3}{S_{13}} \frac{du/dz}{dT/dz}$ is found to be approximately 1.2. Comparison of the definition of a with that of Pr shows that $Pr = 1/a$ so that empirically $Pr = 0.83$ for the stable case. Thus, good agreement of the vertical heat-flux equation with data is shown. The factor of 20 between the measured and the calculated values of \hat{r} indicates that additional work is needed in the modeling of the temperature-variance equation.

NEARLY NEUTRAL ATMOSPHERE

The case is now considered for which the heat flux Q is small and the influence of Q on the turbulent motions is small. Inspection of equations (18) and (20) shows that this case requires that

$$\frac{g|Q|}{T} \ll s \frac{du}{dz} \quad (61a)$$

and from equation (21)

$$\frac{\rho g}{T} |q_1| \ll s \frac{\sqrt{K}}{\Lambda} \quad (61b)$$

For these conditions, S_{ij} is essentially unaffected by Q and is given by the solution for the neutrally stable ground layer and Λ is given by equation (39). Equations (15), (16), and (17) can now be solved for r , q_1 , and dT/dz by assuming series solutions

$$\Lambda \frac{dT}{dz} = \left(\frac{\text{Pr} \sqrt{K} Q}{\rho S_{33}} \right) \sum_{n=0}^{\infty} A_n \left(\frac{z}{L_2} \right)^n \quad (62a)$$

$$r = \left(\frac{\text{Pr} \sqrt{K} Q}{\rho S_{33}} \right)^2 \sum_{n=0}^{\infty} B_n \left(\frac{z}{L_2} \right)^n \quad (62b)$$

$$q_1 = \sum_{n=0}^{\infty} C_n \left(\frac{z}{L_2} \right)^n \quad (62c)$$

Here

$$L_2 = \frac{\rho S_{33} T \sqrt{K}}{g c Q \text{Pr}} \quad (63)$$

is a parameter with the dimension of length, which appears in the analysis, and is quite similar to the Monin-Obukhov length. Equations (62a) and (62b) are substituted into equation (15) so that the B_n are expressed in terms of the A_n . Thus

$$B_n = \left(\frac{2S_{33}}{c^2 K} \right) \frac{A_n}{n^2 c^2 - 2b\text{Pr}^2} \quad (n = 0, 1, 2, \dots) \quad (64)$$

This result is used in equation (17) so that the A_n are found to be given by

$$A_0 = -(1 + 2b) \quad (65a)$$

$$A_{n+1} = \frac{2A_n}{n^2 c^2 - 2bPr^2} \quad (n = 0, 1, 2, \dots) \quad (65b)$$

The temperature gradient is thus

$$\frac{dT}{dz} = - \frac{Pr \sqrt{K} Q (1 + 2b)}{\rho S_{33} c} \left[\frac{1}{z} - \frac{1}{(1 + 2b)L_2} \sum_{n=1}^{\infty} A_n \left(\frac{z}{L_2} \right)^{n-1} \right] \quad (66)$$

and the temperature profile by integration of equation (66) is

$$T = T_0 - \frac{Pr \sqrt{K} Q (1 + 2b)}{\rho S_{33} c} \left[\ln z - \frac{1}{(1 + 2b)} \sum_{n=1}^{\infty} \frac{A_n}{n} \left(\frac{z}{L_2} \right)^n \right]_{z_0}^z \quad (67)$$

Equations (64), (65), and (62b) give

$$r = \frac{Q^2}{\rho^2 S_{33}} \left(\frac{1 + 2b}{b} + 2Pr^2 \sum_{n=1}^{\infty} \frac{A_n \left(\frac{z}{L_2} \right)^n}{n^2 c^2 - 2bPr^2} \right) \quad (68)$$

Equation (16) can now be solved for the horizontal heat flux, thus giving

$$q_1 = \frac{u_*^2 Q}{\rho S_{33}} \left[-1 - \frac{c S_{33}}{k (1 + 2b) Pr u_* \sqrt{K}} + Pr^2 \sum_{n=1}^{\infty} \frac{A_n}{c^2 n^2 - (1 + 2b) Pr^2} \left(\frac{z}{L_2} \right)^2 \right] \quad (69)$$

For $n^2 c^2 \gg 2bPr^2$ (or $n \gg 1$), equation (65b) results in

$$A_n \propto \frac{1}{(n!)^2}$$

so that the series in equations (66) to (69) converge absolutely for all z , and quite rapidly for computations. However, the solution was based on the approximation that heat flux

does not influence the turbulence, as expressed by inequalities (61a) and (61b). By equation (69), $|q_1| \approx u_* |Q| / \rho \sqrt{K}$; therefore, with the use of equations (37) and (38), it is seen that the inequality (61a) implies (61b); thus

$$\frac{\rho g}{T} |q_1| \approx \frac{g|Q|}{T} \frac{u_*}{\sqrt{K}} \approx \frac{g|Q|}{T} \left(\frac{b}{3}\right)^{1/4} < \frac{g|Q|}{T} \ll s \frac{du}{dz} \approx s \frac{\sqrt{K}}{\lambda}$$

Equations (30), (35), and (37) are used to rewrite the right-hand side of inequality (61a); thus

$$\frac{g|Q|}{T} \ll \frac{3b\rho\sqrt{K} S_{33}}{cz}$$

This result is rearranged and equation (63) used, thereby leading to the condition for validity of the solutions of this section that

$$\frac{z}{|L_2|} \ll 3b$$

For this reason, only the leading terms in equations (66) to (69) need be retained.

From equation (66), for $z/L_2 \rightarrow 0$

$$\frac{z}{T_*} \frac{dT}{dz} = \left[\frac{\sqrt{\tilde{K}} k (1 + 2b)}{\hat{S}_{33} c} \right] \text{Pr} = \text{Pr} \quad (70)$$

where equations (35), (37), and (40) have been used to reduce the expression within the brackets to unity. Measurements of the surface layer (ref. 4) show that for $z/L_2 \rightarrow 0$, $\frac{z}{T_*} \frac{dT}{dz} = 0.74$. Thus, $\text{Pr} = 0.74$ for the nearly neutral surface layer. This result agrees fairly well with the value of 0.99 derived for the stable case. From equation (68), for $z/L_2 \rightarrow 0$

$$r = \frac{Q^2}{\rho^2 S_{33}} \frac{1 + 2b}{b} = \frac{k^2 T_*^2}{\hat{S}_{33}} \left(\frac{1 + 2b}{b} \right) \quad (71)$$

From reference 1, measurements show that $r/T_*^2 \approx 1$ for a nearly neutral atmosphere. For $b = 0.0222$, equations (71) and (35) give a value of $r/T_*^2 = 1.49$, which is fair agreement.

FREE CONVECTION

The case is now considered of a surface layer with a strong upward flux of heat and low shear. In this condition, called free convection, the turbulence is primarily thermally produced (ref. 1) in contrast to the condition in which turbulence is produced by shear. Thus, in equation (18) the term $s \frac{du}{dz}$ is neglected, and the turbulence is driven by the term gQ/T in equation (20).

A solution form

$$\sqrt{K} = \beta z^\nu \quad (72a)$$

$$\Lambda = cz^\nu \quad (72b)$$

is assumed, and a solution to equations (18), (19), and (20) is found to be

$$\Lambda = cz \quad (73)$$

$$\sqrt{K} = F_1 \left(\frac{gQ}{T\rho} \right)^{1/3} z^{1/3} \quad (74)$$

$$S_{33} = F_1^2 F_2 \left(\frac{gQ}{T\rho} \right)^{2/3} z^{2/3} \quad (75)$$

$$S_{11} = S_{22} = \frac{F_1^2}{3 \left(1 + 2b - \frac{2}{3} c^2 \right)} \left(\frac{gQ}{T\rho} \right)^{2/3} z^{2/3} \quad (76)$$

where

$$F_1 = \left[\frac{2c}{\left(1 + 2b - \frac{10}{3} c^2 \right) F_2 - \frac{1}{3}} \right]^{1/3} \quad (77)$$

$$F_2 = 1 - \frac{2}{3 \left(1 + 2b - \frac{2}{3} c^2 \right)} \quad (78)$$

With Λ and K determined, equations (15) and (17) may be solved simultaneously for dT/dz and r , thereby resulting in

$$\frac{dT}{dz} = -F_3 \left(\frac{T}{g} \right)^{1/3} \left(\frac{Q}{\rho} \right)^{2/3} z^{-4/3} \quad (79)$$

$$r = F_4 \left(\frac{T}{g} \right)^{2/3} \left(\frac{Q}{\rho} \right)^{4/3} z^{-2/3} \quad (80)$$

where

$$F_3 = \frac{\frac{Pr}{c} (1 + 2b) F_1^2}{F_1^3 F_2 - \frac{c}{9Pr} - bPr} \quad (81)$$

$$F_4 = \frac{c}{bPr - \frac{c^2}{9Pr}} \frac{F_3}{F_1} \quad (82)$$

The shear du/dz and the horizontal heat flux q_1 can now be computed by solving equations (16) and (21). The result is

$$\frac{du}{dz} = F_5 \frac{s}{\rho} \left(\frac{\rho T}{Qg} \right)^{1/3} z^{-4/3} \quad (83)$$

$$q_1 = F_6 \frac{s}{\rho} \left(\frac{Q}{\rho} \right)^{1/3} \left(\frac{T}{g} \right)^{2/3} z^{-2/3} \quad (84)$$

where

$$F_5 = \frac{1 + 2b}{c F_1 F_2} + \frac{F_6}{F_1^2 F_2} \quad (85)$$

$$F_6 = \frac{\frac{1 + 2b}{c F_1 F_2} + F_3}{F_1 \left[\frac{2}{9} \frac{c}{Pr} - \frac{(1 + 2b)}{c} Pr - \frac{1}{F_1^3 F_2} \right]} \quad (86)$$

This solution is based on the assumption that the turbulence generated by buoyancy is much larger than that from shear, so that

$$s \frac{du}{dz} \ll \frac{g}{T} Q \quad (87)$$

Equation (83) may be used to get the result that

$$F_5 \left(\frac{s}{\rho} \right)^2 \ll \left(\frac{gQz}{T\rho} \right)^{4/3}$$

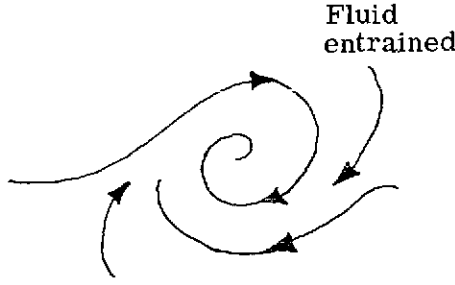
for the free-convection solution to be valid. By use of the Obukhov length L , equation (23), this condition may be written as

$$k F_5^{3/4} \ll \left| \frac{z}{L} \right| \quad (88)$$

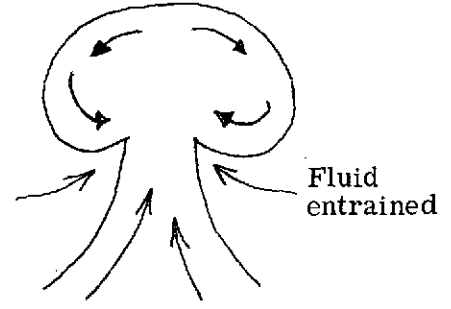
The solutions of this section do not show the arbitrary constants of integration. In obtaining the self-similar solution, it is intrinsically assumed that the effect of the lower boundary has damped out at the altitude required to satisfy condition (88), and that likewise the upper boundary is sufficiently remote that its influence does not diffuse into the region of interest. Thus the terms containing these constants are neglected.

The solutions of this section may be determined by dimensional analysis except for the constants F_1 (refs. 2, 13, and ch. 3 of ref. 1). For this case, u_* is negligible, and the governing variables reduce to Q/ρ , z , and g/T .

If the values $b = 0.0218$ and $c = 0.323$ computed for the neutrally stable case are used in equation (74), \sqrt{K} is computed to be negative. The possibility that \sqrt{K} may be negative is eliminated by consideration of equation (20) for S_{33} . It is concluded that b and c must be quite different for thermally produced turbulence from their values for mechanically produced turbulence. When the structure of the turbulence for the two cases is considered, this difference is not surprising. For the mechanically produced turbulence of neutrally stable or stable turbulent flow, the turbulence appears as eddies, which have essentially a two-dimensional rolling motion in the direction of the flow. For free convection, the turbulence elements have the familiar mushroom-cloud configuration, being axisymmetric about a vertical axis. Mechanically produced turbulence entrains fluid along a sheet, which may be visualized as being rolled up by the eddy. For thermally produced turbulence, fluid is entrained along a ring at the base of the mushroom-cloud top. These concepts are shown in the following sketches:



Mechanically produced
turbulence



Thermally produced
turbulence

Because of this distinction in structure, thermally produced turbulence is very effective in transferring heat, but not momentum. Reynolds analogy does not apply here (ref. 14); therefore, the turbulent Prandtl number may be quite different. The computation of a suitable b , c , and Pr is now considered.

There are some basic requirements for b and c . First, the mixing length Λ must be positive; thus

$$c > 0 \quad (89)$$

Next, $S_{11} = S_{22} > 0$, which by use of equation (76) implies

$$b > \frac{1}{3}c^2 - \frac{1}{2} \quad (90)$$

A more stringent constraint is given by the requirement that $S_{33} > 0$, which by use of equations (75) and (78) implies

$$b > \frac{1}{3}c^2 - \frac{1}{6} \quad (91)$$

Finally, it has been noted that $\sqrt{K} > 0$. This requirement implies, by equation (74), that $F_1 > 0$; thus after some manipulations

$$c^2 < \frac{3}{8} \frac{\delta(\delta - 1)}{\delta - \frac{2}{3}} \quad (92)$$

where

$$\delta = 1 + 2b - \frac{2}{3}c^2 \quad (93)$$

These constraints are shown in figure 1. The condition (92) is applied by determining the boundary in the δ - c plane, and then by mapping this line into the b - c plane of figure 1, which shows the regions of permissible values of b and c . It is seen that for $c > 0$, the requirement that $\sqrt{K} > 0$ is more stringent than the requirements that S_{11} , S_{22} , and S_{33} be positive. Figure 1 also shows that the point b, c for mechanically produced turbulence lies well outside the region of permissible values for purely thermally induced turbulence.

Three numbers, b , c , and Pr are to be computed from experimental data. Many possible conditions may be used. The conditions used here are that the theory and experimental data match for (1) vertical component of turbulent energy, (2) temperature gradient, and (3) temperature variance. These three criteria were chosen because they have been most thoroughly investigated (e.g., ref. 15). Alternate criteria would be to match the turbulent kinetic energy K or the velocity gradient du/dz .

The vertical component of turbulent energy S_{33} is given by equation (75), which may be rewritten as

$$S_{33} = B^2 \left(\frac{gQ}{T\rho} \right)^{2/3} z^{2/3} \quad (94)$$

where

$$F_1^2 F_2 = B^2 \quad (95)$$

This notation agrees with that of reference 1. (Note that $S_{33} = \sigma_w^2$.) Also, this form agrees with that of reference 15, with the notational change $a_2 = B$. From reference 15, $B = 1.4$ (using $k = 0.35$) is found to be the value best fitting experimental data. With this number for B , equation (95) constitutes an implicit relation between b and c , which is shown as a curve in figure 1.

The matching of the temperature gradient is now considered. In reference 4, the nondimensional temperature gradient ϕ_h is shown for very unstable conditions. This gradient can be written approximately as

$$\phi_h \approx 0.23 \left(-\frac{z}{L} \right)^{-1/3} \quad (96)$$

Comparison of equation (96) with equation (79) shows that agreement between these two equations requires that

$$F_3 = 0.23k^{-4/3} = 0.93 \quad (97)$$

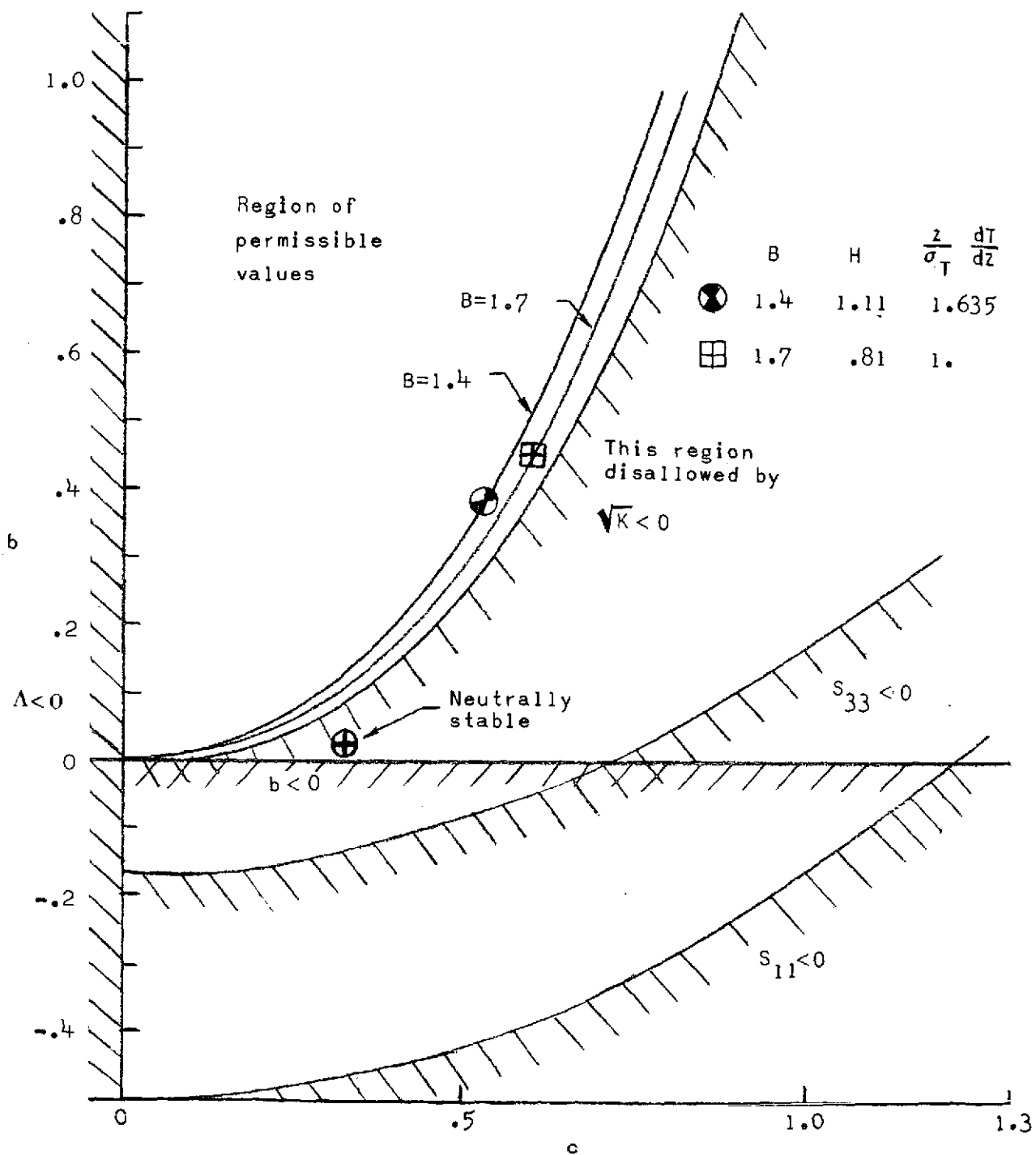


Figure 1.- Values of parameters b and c and regions of permissible values for free convection.

Because F_3 is a function of b , c , and Pr as given by equation (81), equation (97) is an implicit relation among these three parameters.

Finally, the temperature variance is considered. Comparison of equation (80) with the experimental correlations of reference 15 for temperature variance shows that

$$F_4 = a_3^2 \quad (98)$$

By using values from reference 15, the right-hand side of equation (98) is found to be 1.63. As before, equation (98) provides an implicit relation among b , c , and Pr .

Equations (95), (97), and (98) may be solved for b , c , and Pr , thereby resulting in

$$b = 0.39$$

$$c = 0.52$$

$$Pr = 0.56$$

The values for b and c are plotted in figure 1. It is seen that the values for b and c differ drastically from the neutrally stable cases. Also, Pr is considerably less for the unstable case than for the stable case, corresponding to the fact that the unstable atmosphere is far more efficient at transferring heat than momentum. These differences from the neutrally stable and stable cases are attributed to the different structures of mechanically produced and thermally produced turbulence.

As a check on this set of parameters, values from other sources were used. Chapter 4 of reference 1 quotes B as being tentatively 1.7 and shows data indicating that

$$\frac{z}{\sigma_T} \frac{dT}{dz} = 1 \quad (99)$$

for the unstable regime. In reference 13, Priestley gives the result

$$H = \frac{Q}{\rho \left(\frac{g}{T}\right)^{1/2} \left(-\frac{dT}{dz}\right)^{3/2} z^2} = 0.81 \quad (100)$$

Thus, from equation (100) it follows that

$$F_3 = H^{-2/3} \quad (101)$$

which is in the form of equation (97). Equation (99) leads to the relation

$$\frac{F_3^2}{F_4} = \left(\frac{z}{\sigma_T} \frac{dT}{dz} \right)^2 = 1 \quad (102)$$

Numerical solution of equation (95) with $B = 1.7$ together with equations (101) and (102) leads to the result

$$b = 0.45$$

$$c = 0.59$$

$$Pr = 0.65$$

The values for b and c are plotted in figure 1.

In concluding this discussion of the unstable case, it is pointed out that the applicability of the self-similar solution $-\frac{dT}{dz} \propto z^{-4/3}$ has been questioned (ref. 4). In reference 15 the vertical velocity and temperature variances are found to agree with the free-convection form. In order for the self-similar solution to be applicable, the upper boundary must be high enough to insure that the motion is not influenced.

CONCLUDING REMARKS

The turbulence equations as written by Donaldson using the method of invariant modeling are applied to the following limiting cases of the surface or constant flux layer of the planetary boundary layer:

- (1) Neutrally stable
- (2) Stable (above influence of surface roughness)
- (3) Nearly neutrally stable
- (4) Very unstable (free convection)

For the neutrally stable case, the equations are shown to admit as a solution the familiar logarithmic profile. By use of this result, boundary conditions suitable for the surface layer are defined and are simple to apply to rough surfaces.

Expressions for the macroscale length Λ are given for each case. The parameters b , relating the microscale to the macroscale, and c , the ratio of macroscale length to height, are computed to fit atmospheric data. Owing to the structural difference between

mechanically and thermally produced turbulence, different values are found for both b and c for the neutrally stable and unstable cases. The b -value for the stable case agrees very closely with that for the neutrally stable case.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., June 11, 1973.

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